This exam is worth 15% of your final grade (375/2500 points).

There are 7 problems.

You have until 11:15 to finish.

You may use the crib sheet I’ve provided, and calculators should be unnecessary.

Be sure to show your work, as I’ll be giving partial credit.
(1) (50 points) Write an expression containing at most one negation that’s logically equivalent to

\[ \sim \forall x \sim \exists y \sim \forall z \sim M(x, y, z) \]

**Answer:** This simplifies to

\[ \exists x \sim \sim \exists y \sim \forall z \sim M(x, y, z) \]

which simplifies to

\[ \exists x \exists y \sim \forall z \sim M(x, y, z) \]

which simplifies to

\[ \exists x \exists y \exists z \sim M(x, y, z) \]

which simplifies to our answer

\[ \exists x \exists y \exists z M(x, y, z) \]

There’s really only one solution to this (since the negations all cancel out).
(2) (50 points) Provide a proof for the statement $M$ given the following axioms. Be sure to explicitly state the steps of your proof:

(a) $X \implies Y$
(b) $Z \implies M$
(c) $Y \implies M$
(d) $X \lor Z$

**Answer:** We can do proof by division into cases.

e: $X \implies M$ (hypothetical syllogism from $a$ and $c$)
f: $M$ (proof by division into cases using $e$, $b$, and $d$)

There are several other ways to prove this.
(3) (50 points) Vlad Urr, the Count of Kompleck City, makes the claim that

“Someone’s a citizen of Kompleck City if and only if they’re an accountant.”

In terms of accountants (and non-accountants) and Kompleck citizens (and people who aren’t citizens of Kompleck City), what types of people could we encounter whose existence would counter the statement of Count Urr?

**Answer:** There are 2 types we could encounter to counter Count Urr: People who are citizens of Kompleck City, but not accountants, and people who are accountants that aren’t citizens of Kompleck City.
(4) (50 points) (Fill in the blanks):

If \( f = \{(A, B), (B, C), (C, D), (D, A)\} \),

then \( f \circ f \circ f \circ f = \{(A, A), (B, B), (C, C), (D, D)\} \).

**Answer:** (See above.) We can get this by cranking through the computation:

E.g., \( f(f(f(f(A)))) = f(f(f(B))) = f(f(C)) = f(D) = A \).

Another way to do this would be to draw 5 columns of \( A, B, C, \) and \( D \), and connecting \( A \) to \( B \) in the next column, then following \( A \) in the 1st column back to \( A \) in the 5th column.
(5) (50 points total)

- (25 points) Convert $11010101_2$ to decimal (or give me a formula I can plug into my calculator to get the answer).

**Answer:**

$11010101_2 = 2^7 + 2^6 + 2^4 + 2^2 + 2^0 = 128 + 64 + 16 + 4 + 1 = 213$.

- (25 points) What is $111100_2 + 101010_2$ in binary?

**Answer:** We can use the binary addition algorithm that I demonstrated in class:

```
  111  (carries)
 111100
+ 101010
--------
1100110
```

So our answer is $1100110_2$. 

(6) (50 points) Prove the following by induction:

\[ 9 + 9^2 + 9^3 + \cdots + 9^n = \frac{9^{n+1} - 9}{8} \]

(Hint, note that \(9^{n+2} = 9 \cdot 9^{n+1} = (8 + 1)9^{n+1} = 8 \cdot 9^{n+1} + 9^{n+1}.\))

**Answer:** As in any proof by induction, we demonstrate that the property holds for a base case: \(\frac{9^{1+1} - 9}{8} = \frac{81 - 9}{8} = 9 = 9^1.\)

Now, we need our inductive step. We assume that \(9 + 9^2 + 9^3 + \cdots + 9^n = \frac{9^{n+1} - 9}{8},\) and then use this to show that \(9 + 9^2 + 9^3 + \cdots + 9^n + 9^{n+1} = \frac{9^{(n+1)+1} - 9}{8}.\)

We can do this by the following (using the hint provided)

\[
\begin{align*}
(1) \quad \frac{9^{(n+1)+1} - 9}{8} &= \frac{9^{n+2} - 9}{8} \\
(2) &= \frac{8 \cdot 9^{n+1} + 9^{n+1} - 9}{8} \\
(3) &= \frac{8 \cdot 9^{n+1}}{8} + \frac{9^{n+1} - 9}{8} \\
(4) &= \cdot 9^{n+1} + \frac{9^{n+1} - 9}{8} \\
(5) &= \cdot 9^{n+1} + 9 + 9^2 + 9^3 + \cdots + 9^n \\
(6) &= \cdot 9 + 9^2 + 9^3 + \cdots + 9^n + 9^{n+1}
\end{align*}
\]
The following concerns a foot race in which the participants are the Tortoise, Achilles, and the Genie. The race course is $n$ rods in length from the starting line to the goal flag. Each contestant employs a different strategy:

- The Tortoise simply plods along directly to the goal flag.
- Achilles, who can run 10 times as fast as the Tortoise, decides to give the Tortoise an edge by completing the race in the following manner: He dashes the first rod of the race, then back to the starting line, then he dashes 2 rods toward the goal, then back to the starting line, then he dashes 3 rods and back, and continues in this manner until he reaches the goal flag.
- The Genie chants “Sim sim”, the distance to the goal (using our familiar Arabic numerals), then finishes his incantation with “salabim!”, and presto, he’s at the goal. For example, the Genie might say “Sim sim, four two two seven, salabim!”, and he’d be 4,227 rods closer to the goal.

What are the big $O$ runtimes (i.e., the time to reach the goal) in terms of the distance to the goal, $n$ for:

- (25 points) The Tortoise?

**Answer:** The Tortoise’s runtime is linear with the distance to the goal, or $O(n)$.

- (25 points) Achilles?

**Answer:** Achilles will need to run along a distance of $1 + 1 + 2 + 2 + 3 + 3 + \cdots + (n - 1) + (n - 1) + n$ rods. Thus, the length of Achilles’ run will be $\frac{n^2 - n}{2} + n$ rods. Since Achilles’ time is proportional to this distance, his runtime will be quadratic or $O(n^2)$.

- (25 points) The Genie?

**Answer:** The Genie simply needs to intone the distance to the goal. Since the Genie’s using Arabic numerals, he will need to intone $\lceil \log_{10} n \rceil$ numerals (e.g., $\lceil \log_{10} 4227 \rceil = 4.$). The Genie also needs to take the time to say “Sim sim” and “salabim!”, which is constant. The the Genie’s runtime is $O(\log n)$. 
• (25 BONUS points) The Genie, if he chanted the distance using Roman numerals instead of Arabic numerals?

**Answer:** The largest Roman numeral is $\text{M}$, for a thousand. So, if the goal were 10,000 rods away, the Genie would need to say

“Sim sim $\text{M M M M M M M M M}$ salabim!”

and if the goal were 20,000 rods away, the Genie would have to spend almost twice as long on his incantation of

“Sim sim $\text{M M M M M M M M M M M M M M M M}$ salabim!”

So the Genie’s runtime would be linear or $O(n)$.

(There’s also a way to use lines around Roman Numerals to represent larger numbers. (For more on this, see [http://en.wikipedia.org/wiki/Roman_numerals](http://en.wikipedia.org/wiki/Roman_numerals)). However, the Genie couldn’t intone these as easily as he could the Arabic or standard Roman numerals, and the Meta-Genie might get confused.)