(1) In Marc’s casino, I have a game called Mediocre Poker. In this game, I shuffle a deck and (fairly) deal you 5 cards. I take the highest value of your hand and I the pay out one of the following: Pair or less (Nothing), 2 Pair $2, 3 of a kind $5, Straight $25, Flush $50, Full House $100, 4 of a kind $500, Straight flush $10,000, Royal Flush $100,000. (Remember from the class handout that there are 123,552 ways to get a 2 Pair, 54,912 possible 3 of a kinds, 9,180 Straights, 5,112 Flushes, 3,744 Full Houses, 624 possible 4 of a kinds, 32 Straight flushes, and 4 Royal Flushes.)

2598960 2,401,800

123552 . + 54912 . + 9180 . + 5112 . + 3744 . + 624 . + 32 . + 4

(a) What is the expected value of playing one game of Mediocre Poker?

Answer: If we note that there are $\binom{52}{5} = 2,598,960$ possible poker hands. We can get the expected value by a direct application of our formula:

\[ E[X] = 2P(2\text{ Pair}) + 5P(3\text{ of a kind}) + 25P(\text{Straight}) \]

\[ +50P(\text{Flush}) + 100P(\text{Full House}) + 500P(4\text{ of a kind}) \]

\[ +10,000P(\text{Straight flush}) + 100,000P(\text{Royal Flush}) \]

\[ = \frac{2 \times 123,552}{2,598,960} + \frac{5 \times 54,912}{2,598,960} + \frac{25 \times 9,180}{2,598,960} \]

\[ + \frac{50 \times 5,112}{2,598,960} + \frac{100 \times 3,744}{2,598,960} + \frac{500 \times 624}{2,598,960} \]

\[ + \frac{10,000 \times 32}{2,598,960} + \frac{100,000 \times 4}{2,598,960} \]

\[ \approx 0.92851 \]

(b) What is the variance of playing one game of Mediocre Poker?
Answer: Another direct application of our definition of variance:

\[
E \left[ (X - E[X])^2 \right]
\]

\[
= (\$0 - E[X])^2 P(\text{Pair or Less (aka None of the below)})
\]

\[
+ (\$2 - E[X])^2 P(2 \text{ Pair}) + (\$5 - E[X])^2 P(3 \text{ of a kind})
\]

\[
+ (\$25 - E[X])^2 P(\text{Straight}) + (\$50 - E[X])^2 P(\text{Flush})
\]

\[
+ (\$100 - E[X])^2 P(\text{Full House}) + (\$500 - E[X])^2 P(4 \text{ of a kind})
\]

\[
+ (\$10,000 - E[X])^2 P(\text{Straight flush})
\]

\[
+ (\$100,000 - E[X])^2 P(\text{Royal Flush})
\]

\[
= (\$0 - E[X])^2 \frac{2,401,800}{2,598,960}
\]

\[
+ (\$2 - E[X])^2 \frac{123,552}{2,598,960} + (\$5 - E[X])^2 \frac{54,912}{2,598,960}
\]

\[
+ (\$25 - E[X])^2 \frac{9,180}{2,598,960} + (\$50 - E[X])^2 \frac{5,112}{2,598,960}
\]

\[
+ (\$100 - E[X])^2 \frac{3,744}{2,598,960} + (\$500 - E[X])^2 \frac{624}{2,598,960}
\]

\[
+ (\$10,000 - E[X])^2 \frac{32}{2,598,960}
\]

\[
+ (\$100,000 - E[X])^2 \frac{4}{2,598,960}
\]

\[
\approx 8^2 16,703.4443
\]

Note that the variance would be in units of square dollars. (I don’t know what that means except in a mathematical sense.) The standard deviation would be

\[\sqrt{\varepsilon[X]} \approx \$129.24.\]

(2) In Marc’s Casino, I’ve decided to automate Mediocre Poker using a coin operated machine. My machine accepts nickels, dimes, and quarters.

(a) Set up a recurrence relation for the number of different ways one can pay \(5n\) cents to the machine (where the order in which the coins are inserted matters).

Answer: There is only 1 way we can insert 0 cents, and only 1 way to insert 5 cents (a nickel). There are 2 ways to insert 10 cents (2 nickels or a dime), 3 ways
to insert 15 cents (3 nickels, a nickel followed by a dime, and dime followed by a nickel), and 5 ways to insert 20 cents. (This can be figured by listing out all the possibilities, or by setting up a recurrence relation with just nickels and dimes.) So the base cases of our recurrence relation are: \( a_0 = 1 \) \( a_5 = 1 \) \( a_{10} = 2 \) \( a_{15} = 3 \) \( a_{20} = 5 \).

If \( 5n \) is 25 or above (say 30), then the last coin added can be either a quarter (to 5 cents), a dime (to 20 cents), or a nickel (to 25 cents). So we have all the possibilities that there were starting with a quarter less than \( 5n \) (i.e., \( a_{5n-25} \)), plus all the possibilities starting with a dime less than \( 5n \) (i.e., \( a_{5n-10} \)) plus those starting with a nickel less than \( 5n \).

This means that our recurrence relation would be

\[
a_{5n} = a_{5n-25} + a_{5n-10} + a_{5n-5}
\]

(Note that \( a_m \) is undefined if \( m \) isn’t divisible by 5.)

(b) What characteristic equation would we have to solve to obtain a closed form solution for the above recurrence relation?

**Answer:** If we define \( b_n = a_{5n} \), then our recurrence relation in terms of \( b \) becomes

\[
b_n = b_{n-5} + b_{n-2} + b_{n-1}
\]

So we can set up the characteristic equation for \( b \) by setting

\[
r^n = r^{n-1} + r^{n-2} + r^{n-5}
\]

then dividing by \( r^{n-5} \) and rearranging to get

\[
r^5 - r^4 - r^3 - 1 = 0
\]

Alternatively, we can give the characteristic equation for \( a \) by noting that \( a_n = a_{n-25} + a_{n-10} + a_{n-5} \), so we get

\[
t^n = t^{n-5} + t^{n-10} + t^{n-25}
\]
then dividing by $t^{n-25}$ and rearranging to get
\[ t^{25} - t^{20} - t^{15} - 1 = 0 \]

(c) If it costs $1 to play Mediocre Poker, how many ways are there to feed the machine?

**Answer:** Using the above recurrence relation, we just have to compute $a_{100}$.

(I wrote a short program to do this, but it’d take only a few minutes using a calculator): $a_0 = 1, a_5 = 1, a_{10} = 2, a_{15} = 3, a_{20} = 5, a_{25} = 9, a_{30} = 15, a_{35} = 26, a_{40} = 44, a_{45} = 75, a_{50} = 128, a_{55} = 218, a_{60} = 372, a_{65} = 634, a_{70} = 1,081, a_{75} = 1,843, a_{80} = 3,142, a_{85} = 5,357, a_{90} = 9,133, a_{95} = 15,571,$ and $a_{100} = 26,547$.

So our answer is 26,547 ways.

(d) How many ways would there be to insert 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 cents, respectively, if my Mediocre Poker machine took only 3-cent-pieces and pennies?

**Answer:** Following similar reasoning to part a, we can set up the recurrence relation: $a_0 = 1, a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-3}$. So we get that the 0th through 10th terms are: 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, and 28 ways to insert 0 through 10 cents.

(3) (a) List all 16 of the relations on \{0, 1\}:

**Answer:** In no particular order, they are:

(i) \{(1, 1)\}

(ii) \{(1, 0) (0, 1) (0, 0)\}

(iii) \{(1, 1) (0, 1) (0, 0)\}

(iv) \{(1, 1) (1, 0) (0, 0)\}

(v) \{(1, 1) (0, 0)\}

(vi) \{(0, 1) (0, 1)\}

(vii) \{(1, 1) (0, 0)\}

(viii) \{(0, 0)\}
(ix) \{(1, 1) (1, 0) (0, 1) (0, 0)\}

(x) \{\}

(xi) \{(1, 1) (1, 0)\}

(xii) \{(1, 1) (0, 1)\}

(xiii) \{(0, 1) (0, 0)\}

(xiv) \{(1, 0) (0, 0)\}

(xv) \{(0, 1)\}

(xvi) \{(1, 0)\}

(b) **BONUS: Name** all of the 16 relations on \{\textit{True, False}\} (e.g., “and”, “only if”, “nand”):

**Answer:** Abbreviating \(T\) for \textit{True} and \(F\) for false, the 16 relations are

- \textit{and}: \{(T, T)\}
- \textit{nand}: \{(T, F) (F, T) (F, F)\}
- \textit{only if (or implies)}: \{(T, T) (F, T) (F, F)\}
- \textit{if}: \{(T, T) (T, F) (F, F)\}
- \textit{if and only if}: \{(T, T) (F, F)\}
- \textit{xor}: \{(T, F) (F, T)\}
- \textit{or}: \{(T, T) (T, F) (F, T)\}
- \textit{nor}: \{(F, F)\}
- \textit{true (or tautology)}: \{(T, T) (T, F) (F, T) (F, F)\}
- \textit{false (or contradiction)}: \{\}
- “the former”: \{(T, T) (T, F)\}
- “the latter”: \{(T, T) (F, T)\}
- “not the former”: \{(F, T) (F, F)\}
- “not the latter”: \{(T, F) (F, F)\}
- \textit{nif (not if)}: \{(F, T)\}
- \textit{nimp (not implies)}: \{(T, F)\}